
**FIELD OF BOREL GROUP INVARIANT OF ADJOINT
REPRESENTATION OF THE GROUP $GL(n, K)$ ¹**

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The paper is devoted to invariant theory problems. In particular, to the problem of finding generators of invariant fields in an explicit form. The set of generators is given for invariant field of unitriangular group of adjoint representation of $GL(n, K)$. Moreover, the set of generators of borel group invariant field is constructed and their algebraic independence is proved. Keywords: Lie group, adjoint representation, the field of invariant, generators of the field of invariant, Borel group.

1. Preliminaries

The main problem of the theory of invariants of algebraic groups is a description of the structure of the ring and the field of invariants. There are two general theorem.

Theorem 1.1. ([1]) Let V be a vector space over the field K . The field of invariants with respect to any triangularizable subgroup of $GL(V)$ is rational. It means, that there are invariants $Q_1, \dots, Q_n \in K(V)$ such that the field of invariants is the field of rational functions $K(Q_1, \dots, Q_n)$.

Definition 1. A rational transformation α of K^n is triangular if in a suitable coordinate system all transformations $\alpha(g), g \in G$, have the form of

$$x_i \rightarrow a_i x_i + f_i(x_{i+1}, \dots, x_n),$$

where $a_i \in K^*, f_i \in k(x_{i+1}, \dots, x_n)$.

Theorem 1.2. ([2]) Let K be an algebraically closed field of characteristic zero. For any subgroup, consisting of rational triangular transformations K^n , a field of invariants is rational.

Now introduce some notation. Let K be a field of characteristic zero, V be a vector space over the field K , $G = GL(n, K)$ is a group of nonsingular matrices, and B is uppertriangular subgroup and U is upperunitriangular subgroup of G . We consider the adjoint action of the group G in the vector space $V = \text{Mat}(n, K) : Ad_g X = gXg^{-1}, g \in G, X \in V$. Let's define representation Ad_g in $K[V]$ (accordingly, $\mathcal{F} = K(V)$) by the formula:

$$Ad_g f(x) = f(Ad_g^{-1}(x)).$$

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Denote \mathcal{F}^U (accordingly \mathcal{F}^B) as subfield of U -invariants (accordingly, B -invariants) in the field \mathcal{F} . From theorems 1.1. and 1.2. the fields \mathcal{F}^U and \mathcal{F}^B are rational.

The main goal of this work is to find generators of the field B -invariants for the adjoint representation in explicit form. The main result will be formulated in the theorem 2.2.

2. The main results.

2.1. Generators of U -invariants field.

Let $x_{i,j}$ be a system of standard coordinate functions on $V = \text{Mat}(n, K)$. Let's construct a system of n minors of the form:

$$J_i = \begin{vmatrix} x_{n-i+1,1} & \cdots & x_{n-i+1,i} \\ \vdots & \cdots & \vdots \\ x_{n,1} & \cdots & x_{n,i} \end{vmatrix}, i = \overline{1, n}. \quad (2.1)$$

Extend this set to the set of invariants of $\frac{n(n+1)}{2}$ elements as follows. We associate with each J_i the system of i determinants $J_{i,j}$, where $0 \leq j \leq i-1$. The determinant $J_{i,0}$ coincides with the minor J_i . The first $i-j$ rows in the determinant $J_{i,j}$, where $1 \leq j \leq i-1$, coincide with the last $i-j$ rows in the minor J_i ; the last j rows in $J_{i,j}$ coincide with the similar rows in the minor $J_i(X^*)$ of the adjoint matrix $X^* = (x_{i,j}^*)$, which is defined as:

$$X \cdot X^* = X^* \cdot X = \det X \cdot E$$

Exapmle 1. The case of $n = 2$,

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad X^* = \begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix},$$

$$J_{1,0} = x_{2,1},$$

$$J_{2,0} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \quad J_{2,1} = \begin{vmatrix} x_{21} & x_{22} \\ x_{21}^* & x_{22}^* \end{vmatrix}.$$

Example 2. The case of $n = 3$,

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad X^* = \begin{pmatrix} x_{11}^* & x_{12}^* & x_{13}^* \\ x_{21}^* & x_{22}^* & x_{23}^* \\ x_{31}^* & x_{32}^* & x_{33}^* \end{pmatrix},$$

$$J_{1,0} = x_{3,1},$$

$$J_{2,0} = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \quad J_{2,1} = \begin{vmatrix} x_{31} & x_{32} \\ x_{31}^* & x_{32}^* \end{vmatrix},$$

$$J_{3,0} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}, \quad J_{3,1} = \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{31}^* & x_{32}^* & x_{33}^* \end{vmatrix}, \quad J_{3,2} = \begin{vmatrix} x_{31} & x_{32} & x_{33} \\ x_{21}^* & x_{22}^* & x_{23}^* \\ x_{31}^* & x_{32}^* & x_{33}^* \end{vmatrix}.$$

Theorem 2.1.(Vyatkina K.A, Panov A.N. [3]) The field of U -invariants of adjoint representation of the group $GL(n, K)$ is the field of rational functions of $\{J_{i,j} : 1 \leq i \leq n, 0 \leq j \leq i-1\}$.

2.2. Generators of B -invariants field.

Let H denote the subgroup of diagonal matrix in $GL(n, K)$. Its easy to see, that adjoint representation of subgroup H preserves the field \mathcal{F}^U , and the field \mathcal{F}^B coincides with the subfield of H -invariants in \mathcal{F}^U .

Definition 2. We call function $f \in k[V]$ cumulant, if

$$Ad_h f = \chi_h(f) \cdot f,$$

where $h \in H$ and $\chi_h(f) \in K^*$. The character $\chi_h(f)$ of subgroup H we call weight of cumulant f .

Let $h = \text{diag}(a_1, \dots, a_n) \in H$. Then $Ad_h x_{i,j} = \frac{a_j}{a_i} x_{i,j}$, that is

$$\chi_h(x_{i,j}) = \frac{a_j}{a_i}.$$

Note, that adjugate matrix X^* changing similary. Indeed, the definition of the adjoint matrix gives

$$X \cdot X^* = X^* \cdot X = \det X \cdot E,$$

$$Ad_h(X) \cdot Ad_h(X^*) = \det(Ad_h(X)) \cdot E.$$

Thence $Ad_h(X^*) = (Ad_h(X))^*$ and hence $Ad_h x_{i,j}^* = \frac{a_j}{a_i} x_{i,j}^*$. Then

$$\chi_h(x_{i,j}^*) = \chi_h(x_{i,j}) = \frac{a_j}{a_i}. \quad (2.2)$$

Let's define some notations:

$$1. y_1 = J_{1,0}, \quad y_i = \frac{J_{i,i-1}}{J_{i-1,0}}, \quad \text{где } 2 \leq i \leq n,$$

$$2. Y_{i,j} = \frac{J_{i,j} \cdot y_{n-i+j+1}}{J_{i-1,j} \cdot y_i}, \quad \text{где } 2 \leq i \leq n, \quad 0 \leq j \leq i-2.$$

Now let's prove the next lemma.

Lemma 1. Let $F = K(z_1, \dots, z_n)$ — field of a rational functions of variables z_1, \dots, z_n from field K . Let diagonal subgroup H act on the field F according to the formula: $hz_i = \chi_h(z_i)z_i$. Then subfield of H -invariants F^H generated by the elements of the form

$$z_1^{m_1} \dots z_n^{m_n}, \quad \text{where } m_i \in \mathbb{Z} \quad \text{and} \quad \chi_h(z_1)^{m_1} \dots \chi_h(z_n)^{m_n} = 1.$$

Definition 3. We call the elementary transformation of a system of generators z_1, \dots, z_n of the field F the transition from the old system to the new system, using the operations like:

$$z_i \rightarrow z_i \cdot z_j^{\pm 1}, \text{ where } i \neq j.$$

Definition 4. We claim that two system of generators are equivalent, if there are elementary transformation, which turn one system to another.

Now let proceed to the statement of the main theorem and prove it.

Theorem 2.2. The field of B -invariants of adjoint representation of the group $GL(n, K)$ is the field of rational functions of

$$\{y_n, Y_{i,j} : 2 \leq i \leq n, 0 \leq j \leq i-2\} \quad (2.3)$$

Proof. As we say above, \mathcal{F}^U is generated by the set of algebraically independent elements $\{J_{1,0}, \dots, J_{n,n-1}\}$ (see Theorem 2.1). We refer to the following table as a table of generators:

$$\begin{bmatrix} & & & & J_{n,0} \\ & & & J_{n-1,0} & J_{n,1} \\ & & \ddots & \vdots & \vdots \\ & J_{2,0} & \dots & J_{n-1,n-3} & J_{n,n-2} \\ J_{1,0} & J_{2,1} & \dots & J_{n-1,n-2} & J_{n,n-1} \end{bmatrix} \quad (2.4)$$

Each of generators $J_{i,j}$ is semi-invariant. Similarly to the table of generators we construct the table of weights:

$$\begin{bmatrix} & & & & \chi_h(J_{n,0}) \\ & & & \chi_h(J_{n-1,0}) & \chi_h(J_{n,1}) \\ & & \ddots & \vdots & \vdots \\ & \chi_h(J_{2,0}) & \dots & \chi_h(J_{n-1,n-3}) & \chi_h(J_{n,n-2}) \\ \chi_h(J_{1,0}) & \chi_h(J_{2,1}) & \dots & \chi_h(J_{n-1,n-2}) & \chi_h(J_{n,n-1}) \end{bmatrix} \quad (2.5)$$

The action of subgroup H on the field \mathcal{F}^U satisfies the conditions of Lemma 1. Starting with the system $\{J_{i,j}\}$ we shell construct by elementary transformation the new system of generators, which has the simplest weighting table, using only elementary transformations.

Apply the formula (2.2), we have

$$\chi_h(J_{i,j}) = \frac{a_1 \dots a_i}{a_n \dots a_{n-j+1} \cdot a_n \dots a_{n-i+j+1}} \quad (2.6)$$

In particular,

$$\begin{aligned} \chi_h(J_{1,0}) &= \frac{a_1}{a_n}, \dots, \chi_h(J_{i,0}) = \frac{a_1 \dots a_i}{a_n \dots a_{n-i+1}}, \dots, \chi_h(J_{n,0}) = 1, \\ \chi_h(J_{2,1}) &= \frac{a_1 a_2}{a_n a_n}, \dots, \chi_h(J_{i,i-1}) = \frac{a_1 \dots a_i}{a_n \dots a_{n-i+2} a_n}, \dots, \chi_h(J_{n,n-1}) = \frac{a_1}{a_n}. \end{aligned}$$

After substitution the obtained values to (2.5), we obtain:

$$\begin{bmatrix} & & & & \frac{a_1 \dots a_{n-1}}{a_n \dots a_2} & \frac{1}{\chi_h(J_{n,1})} \\ & & & \ddots & \vdots & \vdots \\ & & \frac{a_1 a_2 a_3}{a_n a_{n-1} a_{n-2}} & \dots & \chi_h(J_{n-1,n-4}) & \chi_h(J_{n,n-3}) \\ & \frac{a_1 a_2}{a_n a_{n-1}} & \chi_h(J_{3,1}) & \dots & \chi_h(J_{n-1,n-3}) & \chi_h(J_{n,n-2}) \\ \frac{a_1}{a_n} & \frac{a_n a_{n-1}}{a_1 a_2} & \frac{a_1 a_2 a_3}{a_n a_n a_{n-1}} & \dots & \frac{a_1 \dots a_{n-1}}{a_n a_n a_{n-1} \dots a_3} & \frac{a_1}{a_n} \end{bmatrix}$$

Let's make the follow series of elementary transformations.

1. We proceed from system $\{J_{i,j}\}$ to system $\{J'_{i,j}\}$, where $J'_{i,j} = \frac{J_{i,j}}{J_{i-1,j}}$ for $3 \leq i \leq n$, $1 \leq j \leq i-2$; and $J'_{i,j} = J_{i,j}$ for others. Since

$$\chi_h(J'_{i,j}) = \frac{a_i}{a_{n-i+j+1}},$$

the table of weights for $\{J'_{i,j}\}$ takes the form:

$$\begin{bmatrix} & & & & \frac{a_1 \dots a_{n-1}}{a_n \dots a_2} & \frac{1}{a_n} \\ & & & \ddots & \vdots & \vdots \\ & & \frac{a_1 a_2}{a_n a_{n-1}} & \dots & \frac{a_{n-1}}{a_{n-1}} & \frac{a_n}{a_{n-1}} \\ \frac{a_1}{a_n} & \frac{a_n a_{n-1}}{a_1 a_2} & \dots & \frac{a_{n-1}}{a_1 \dots a_{n-1}} & \frac{a_{n-1}}{a_1} & \frac{a_n}{a_n} \end{bmatrix} \quad (2.7)$$

2. Each $J'_{i,i-1}$, where $2 \leq i \leq n$, is replaced by $y_i = \frac{J'_{i,i-1}}{J'_{i-1,0}} = \frac{J_{i,i-1}}{J_{i-1,0}}$. Notice, that $y_1 = J'_{1,0} = J_{1,0}$. The table of generators becomes:

$$\begin{bmatrix} & & & & J'_{n,0} \\ & & & J'_{n-1,0} & J'_{n,1} \\ & & \ddots & \vdots & \vdots \\ & J'_{2,0} & \dots & J'_{n-1,n-3} & J'_{n,n-2} \\ y_1 & y_2 & \dots & y_{n-1} & y_n \end{bmatrix} \quad (2.8)$$

We calculate $\chi_h(y_i)$:

$$\chi_h(y_1) = \frac{a_1}{a_n}, \dots, \chi_h(y_i) = \frac{a_i}{a_n}, \dots, \chi_h(y_{n-1}) = \frac{a_{n-1}}{a_n}, \chi_h(y_n) = \frac{a_n}{a_n} = 1$$

Substituting these weights in (2.7), we obtain:

$$\begin{bmatrix} & & & & 1 \\ & & & \frac{a_1}{a_n} & \frac{a_n}{a_2} \\ & & \ddots & \vdots & \vdots \\ & \frac{a_1 a_2}{a_n a_{n-1}} & \dots & \frac{a_{n-1}}{a_n} & \frac{a_n}{a_{n-1}} \\ \frac{a_1}{a_n} & \frac{a_2}{a_n} & \dots & \frac{a_{n-1}}{a_n} & 1 \end{bmatrix}$$

3. Now we proceed from the system of generators

$$\{y_1, \dots, y_n, J'_{i,j}, 2 \leq i \leq n, 0 \leq j \leq i-2\}$$

to the new system of elements, defined, setting $J''_{i,0} = \frac{J'_{i,0}}{J'_{i-1,0}}$ и $J''_{i,j} = J'_{i,j}$ for the other generators. The weights can be calculated by the formula:

$$\chi_h(J''_{i,0}) = \frac{a_i}{a_{n-i+1}}, \text{ где } 2 \leq i \leq n-1.$$

Table of the weights has the form:

$$\begin{bmatrix} & & & & 1 \\ & & & \frac{a_{n-1}}{a_2} & \frac{a_n}{a_2} \\ & & \ddots & \vdots & \vdots \\ & \frac{a_2}{a_n} & \dots & \frac{a_{n-1}}{a_n} & \frac{a_n}{a_{n-1}} \\ \frac{a_1}{a_n} & \frac{a_2}{a_n} & \dots & \frac{a_{n-1}}{a_n} & 1 \end{bmatrix}$$

4. Each $J''_{i,j}$, where $2 \leq i \leq n, 0 \leq j \leq i-2$, we replace by the

$$Y_{i,j} = \frac{J''_{i,j} \cdot y_{n-i+j+1}}{y_i}.$$

We obtain

$$\chi_h(Y_{i,j}) = \chi_h\left(\frac{J''_{i,j} \cdot y_{n-i+j+1}}{y_i}\right) = \frac{a_i}{a_{n-i+j+1}} \cdot \frac{a_{n-i+j+1}}{a_n} \cdot \frac{a_n}{a_i} = 1,$$

that is, each $Y_{i,j}$ is invariant.

The table of generators and the table of weights has the form:

$$\begin{bmatrix} & & & & Y_{n,0} \\ & & & Y_{n-1,0} & Y_{n,1} \\ & & \ddots & \vdots & \vdots \\ & Y_{2,0} & \dots & Y_{n-1,n-3} & Y_{n,n-2} \\ y_1 & y_2 & \dots & y_{n-1} & y_n \end{bmatrix} \quad \begin{bmatrix} & & & & 1 \\ & & & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & \frac{1}{a_n} & \frac{1}{a_2} & \dots & \frac{1}{a_n} \\ \frac{a_1}{a_n} & \frac{a_2}{a_n} & \dots & \frac{a_{n-1}}{a_n} & 1 \end{bmatrix}$$

Applying Lemma 1 to the system of generators

$$\{y_k, Y_{i,j} : 1 \leq k \leq n-1, 2 \leq i \leq n, 0 \leq j \leq i-2\},$$

we conclude that the set of algebraically independent elements $\{y_n, Y_{i,j} : 2 \leq i \leq n, 0 \leq j \leq i-2\}$ generates the field \mathcal{F}^B .

References

- [1] K.Miyata, “Invariants of certain groups ”, Nagoya Math. J., 1971, т. 1, №41 69-73
- [2] E. B. Vinberg, V. L. Popov, “Invariant theory ”, Algebraic geometry – 4, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 55, VINITI, Moscow, 1989, 137–309
- [3] K.A. Vyatkina, A.N. Panov “Field of U-invariants of adjoint representation of the group $GL(n, K)$ ”, Matematicheskie Zametki, 2013, т. 93, №1, 144–147

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